

EXTREME RESIDUES OF DEDEKIND ZETA FUNCTIONS

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ABSTRACT. In a family of S_{d+1} -fields ($d = 2, 3, 4$), we obtain the true upper and lower bound of the residues of Dedekind zeta functions except for a density zero set. For S_5 -fields, we need to assume the strong Artin conjecture. We also show that there exists an infinite family of number fields with the upper and lower bound, resp.

1. INTRODUCTION

For a quadratic extension $K = \mathbb{Q}(\sqrt{D})$ with a fundamental discriminant D , $Res_{s=1}\zeta_K(s) = L(1, \chi_D)$, where $\chi_D = \left(\frac{D}{\cdot}\right)$ is the quadratic character. In this case, Littlewood [10] obtained the bound

$$\left(\frac{1}{2} + o(1)\right) \frac{\zeta(2)}{e^\gamma \log \log |D|} \leq L(1, \chi_D) \leq (2 + o(1))e^\gamma \log \log |D|$$

under GRH, where γ is the Euler-Mascheroni constant. Under the same hypothesis, he also constructed an infinite family of quadratic fields with $L(1, \chi_D) \geq (1 + o(1))e^\gamma \log \log |D|$ and an infinite family of quadratic fields with $L(1, \chi_D) \leq (1 + o(1))\frac{\zeta(2)}{e^\gamma \log \log |D|}$. Later, Chowla [3] established the latter omega result unconditionally. It has been conjectured that the true upper and lower bounds are $(1 + o(1))e^\gamma \log \log |D|$ and $(1 + o(1))\frac{\zeta(2)}{e^\gamma \log \log |D|}$, resp. In [11], Montgomery and Vaughan considered the distribution of $L(1, \chi_D)$ via random variables which take ± 1 with equal probability. They proposed three conjectures which support the expected bounds. In [5], some of the conjectures were proved by Granville and Soundararajan.

For a number field K of degree $d + 1$, the lower bound and the upper bound of $Res_{s=1}\zeta_K(s)$ under GRH and the strong Artin conjecture for $\zeta_K(s)/\zeta(s)$ are

$$(1.1) \quad \left(\frac{1}{2} + o(1)\right) \frac{\zeta(d+1)}{e^\gamma \log \log |D_K|} \leq Res_{s=1}\zeta_K(s) \leq (2 + o(1))^d (e^\gamma \log \log |D_K|)^d,$$

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where D_K is the discriminant of a number field K . The proof of (1.1) is given in Section 3 since at least the upper bound is well-known but it is hard to find its proof in the literature.

As in the quadratic extension case, we may conjecture that $(1 + o(1))(e^\gamma \log \log |D_K|)^d$ and $(1 + o(1))\frac{\zeta(d+1)}{e^\gamma \log \log |D_K|}$ are the true upper and lower bound, resp. In this paper, we show that it is the case except for a density zero set in a family of number fields. A number field K of degree $d+1$ is called a S_{d+1} -field if its Galois closure over \mathbb{Q} is an S_{d+1} Galois extension. For a S_{d+1} -field K , we have a decomposition of $\zeta_K(s)$:

$$\zeta_K(s) = \zeta(s)L(s, \rho, \widehat{K}/\mathbb{Q}),$$

where \widehat{K} is the Galois closure of K over \mathbb{Q} and ρ is the standard representation of S_{d+1} . For simplicity, we denote $L(s, \rho, \widehat{K}/\mathbb{Q})$ by $L(s, \rho)$. Hence $\text{Res}_{s=1} \zeta_K(s) = L(1, \rho)$. Then, our first main theorem is

Theorem 1.2. *Let $L(X)$ be a set of S_{d+1} -fields with $X/2 \leq |D_K| \leq X$, $d+1 = 3, 4$ and 5 . For S_5 -fields, we assume the strong Artin conjecture for $L(s, \rho)$. Then, except for $O(Xe^{-c' \frac{\log X}{\log \log X} \log \log \log X})$ L -functions for some constant $c' > 0$,*

$$(1 + o(1))\frac{\zeta(d+1)}{e^\gamma \log \log |D_K|} \leq L(1, \rho) \leq (1 + o(1))(e^\gamma \log \log |D_K|)^d.$$

where $o(1) = O\left(\frac{1}{(\log \log |D_K|)^{1/2}}\right)$.

Furthermore, under the same hypothesis, we construct an infinite family of S_{d+1} -fields with extreme residue values.

Theorem 1.3. *Let $d+1 = 3, 4$, and 5 . For $d+1 = 5$, we assume the strong Artin conjecture. Then*

(1) *The number of S_{d+1} -fields K of signature (r_1, r_2) with $\frac{X}{2} \leq |D_K| \leq X$ for which*

$$\begin{aligned} L(1, \rho) &= \prod_{p \leq y} (1 - p^{-1})^{-d} \left(1 + O\left(\frac{1}{(\log \log |D_K|)^{1/2}}\right) \right) \\ &= (e^\gamma \log \log |D_K|)^d \left(1 + O\left(\frac{1}{(\log \log |D_K|)^{1/2}}\right) \right) \end{aligned}$$

$$\text{is } \geq A(r_2)X \exp\left(-\log |S_{d+1}| \cdot \frac{\log X}{\log \log X} - \log \log \log X\right).$$

(2) *The number of S_{d+1} -fields K of signature (r_1, r_2) with $\frac{X}{2} \leq |D_K| \leq X$ for which*

$$L(1, \rho) = \frac{\zeta(d+1)}{e^\gamma \log \log |D_K|} \left(1 + O\left(\frac{1}{(\log \log X)^{1/2}}\right) \right)$$

$$is \geq A(r_2)X \exp \left(-\log \frac{|S_{d+1}|}{(d+1)} \cdot \frac{\log X}{\log \log X} - \log \log \log X \right).$$

We also construct an infinite family of S_{d+1} -fields with bounded residues.

Theorem 1.4. *Let $d+1 = 3, 4$, and 5 . For $d+1 = 5$, we assume the strong Artin conjecture.*

Then the number of S_{d+1} -fields K of signature (r_1, r_2) with $\frac{X}{2} \leq |D_K| \leq X$ for which

$$L(1, \rho) = \begin{cases} \zeta(2)^{\frac{d}{2}}(1 + o(1)), & \text{if } d \text{ is even} \\ \zeta(2)^{\frac{d-3}{2}}\zeta(3)(1 + o(1)), & \text{if } d \geq 3 \text{ is odd.} \end{cases}.$$

$$is \geq A(r_2)X \exp \left(-\log \frac{|S_{d+1}|}{|C|} \cdot \frac{\log X}{\log \log X} - \log \log \log X \right), \text{ where}$$

$$C = \begin{cases} (1, 2)(3, 4) \cdots (d-1, d), & \text{if } d \text{ is even} \\ (1, 2)(3, 4) \cdots (d-4, d-3)(d-2, d-1, d), & \text{if } d \text{ is odd} \end{cases}.$$

This work is motivated by the work of Lamzouri [8, 9], who constructed primitive characters χ with large values of $L(1, \chi)$. Basically, we follow [8, 9, 5, 11]. The arguments in [8] are easily extended. However, obtaining an analogue of Proposition 2.4 in [8] is a main obstacle to extend his method. It is resolved in Proposition 4.3.

2. COUNTING NUMBER FIELDS WITH LOCAL CONDITIONS

Let K be a S_{d+1} -field of signature (r_1, r_2) for $d+1 \geq 3$. We assume that we can count S_{d+1} -fields with finitely many local conditions. Namely, let $\mathcal{S} = (\mathcal{LC}_p)$ be a finite set of local conditions: $\mathcal{LC}_p = \mathcal{S}_{p,C}$ means that p is unramified and the conjugacy class of Frob_p is C . Define $|\mathcal{S}_{p,C}| = \frac{|C|}{|S_n|(1+f(p))}$ for some function $f(p)$ which satisfies $f(p) = O(\frac{1}{p})$. There are also several splitting types of ramified primes, which are denoted by r_1, r_2, \dots, r_w : $\mathcal{LC}_p = \mathcal{S}_{p,r_j}$ means that p is ramified and its splitting type is r_j . We assume that there are positive valued functions $c_1(p), c_2(p), \dots, c_w(p)$ with $\sum_{i=1}^w c_i(p) = f(p)$ and define $|\mathcal{S}_{p,r_i}| = \frac{c_i(p)}{1+f(p)}$. We define the local condition $\mathcal{LC}_p = \mathcal{S}_{p,r}$ which means that p is ramified, i.e, $r = r_j$ for some j . Define $|\mathcal{S}_{p,r}| = \frac{f(p)}{1+f(p)}$. Let $|\mathcal{S}| = \prod_p |\mathcal{LC}_p|$.

Let $L(X)^{r_2}$ be the set of S_{d+1} -fields K of signature (r_1, r_2) with $\frac{X}{2} < |D_K| < X$, and let $L(X; \mathcal{S})^{r_2}$ be the set of S_{d+1} -fields K of signature (r_1, r_2) with $\frac{X}{2} < |D_K| < X$ and the local conditions \mathcal{S} . Then we have

Conjecture 2.1. *For some positive constants $\delta < 1$ and κ ,*

$$(2.2) \quad \begin{aligned} |L(X)^{r_2}| &= A(r_2)X + O(X^\delta), \\ |L(X; \mathcal{S})^{r_2}| &= |\mathcal{S}|A(r_2)X + O\left(\left(\prod_{p \in \mathcal{S}} p\right)^\kappa X^\delta\right), \end{aligned}$$

where the implied constant is uniformly bounded for p and local conditions at p .

It is worth noting here that we can control only all the primes up to $c \log X$, where $c < (1 - \delta)/\kappa$. If we impose local conditions for all $p \leq c' \log X$ with $c' \geq (1 - \delta)/\kappa$, the error term in Conjecture 2.1 would be larger than the size of $L(X)^{r_2}$.

For S_3 -fields, the conjecture was shown by Taniguchi and Thorne [12]. In [2]¹, we proved that Conjecture 2.1 is true for S_4 and S_5 -fields.

3. FORMULA FOR $L(1, \rho)$ UNDER A CERTAIN ZERO-FREE REGION

In this paper, we assume the strong Artin conjecture, namely, the Artin L -function $L(s, \rho)$ is an automorphic representation of GL_d . This is true for S_3 -fields and S_4 -fields. It implies the Artin conjecture, namely, $L(s, \rho)$ is entire. For this section, we only need the Artin conjecture. However, in Section 4, we need the strong Artin conjecture in order to use Kowalski-Michel zero density theorem [7]. We find an expression of $L(1, \rho)$ as a product over small primes under assumption that $L(s, \rho)$ has a certain zero-free region. Here all the implicit constants only depend on the degree d of $L(s, \rho)$.

For $\operatorname{Re}(s) > 1$, $L(s, \rho)$ has the Euler product:

$$L(s, \rho) = \prod_p \prod_{i=1}^d \left(1 - \frac{\alpha_i(p)}{p^s}\right)^{-1}.$$

Then, for $\operatorname{Re}(s) > 1$,

$$\log L(s, \rho) = \sum_{n=2}^{\infty} \frac{\Lambda(n) a_\rho(n)}{n^s \log n},$$

where $a_\rho(p^k) = \alpha_1(p)^k + \cdots + \alpha_d(p)^k$. First, we show that when $L(s, \rho)$ has a certain zero-free region, the value $\log L(1, \rho)$ is determined by a short sum.

¹In [2], we used the Greek letter γ in place of κ . However, γ is taken for the Euler-Mascheroni constant in this article.

Proposition 3.1. *If $L(s, \rho)$ is entire and is zero-free in the rectangle $[\alpha, 1] \times [-x, x]$, where $x = (\log N)^\beta$, $\beta(1 - \alpha) > 2$, and N is the conductor of ρ , then*

$$(3.2) \quad \log L(1, \rho) = \sum_{n < x} \frac{\Lambda(n) a_\rho(n)}{n \log n} + O((\log N)^{-1}).$$

Proof. By Perron's formula,

$$\frac{1}{2\pi i} \int_{c-ix}^{c+ix} \log L(1+s, \rho) \frac{x^s}{s} ds = \sum_{n < x} \frac{\Lambda(n) a_\rho(n)}{n \log n} + O\left(\frac{\log x}{x}\right).$$

where $c = \frac{1}{\log x}$.

Now move the contour to $\operatorname{Re}(s) = \alpha - 1 + \frac{1}{\log x}$. We get the residue $\log L(1, \rho)$ at $s = 0$. So the left hand side is $\log L(1, \rho)$ plus

$$\frac{1}{2\pi i} \left(\int_{c-ix}^{\alpha-1+c-ix} + \int_{\alpha-1+c-ix}^{\alpha-1+c+ix} + \int_{\alpha-1+c+ix}^{c+ix} \right) \log L(1+s, \rho) \frac{x^s}{s} ds.$$

In order to estimate $|\log L(s, \rho)|$ for $\alpha + c \leq \operatorname{Re}(s) \leq 1 + c$, we follow [6, Lemma 8.1]: Consider the circles with centre $2 + it$ and radii $r = 2 - \sigma < R = 2 - \alpha$. By the assumption, $\log L(s, \rho)$ is holomorphic inside the larger circle. By Daileida [4, page 222], for $\frac{1}{2} < \operatorname{Re}(s) \leq \frac{3}{2}$, $|L(s, \rho)| \leq N^{\frac{1}{2}}(|s| + 1)^{\frac{d}{2}}$. Hence $\operatorname{Re} \log L(s, \rho) = \log |\log L(s, \rho)| \ll \log N + \log(|s| + 1)$. Clearly, if $\operatorname{Re}(s) \geq \frac{3}{2}$, $|\log L(s, \rho)| = O(1)$. By the Borel-Carathéodory theorem,

$$|\log L(s, \rho)| \leq \frac{2r}{R-r} \max_{|z-(2+it)|=R} \operatorname{Re} \log L(z, \rho) + \frac{R+r}{R-r} |\log L(2+it, \rho)| \ll (\log x)(\log N + \log(|s| + 1)).$$

Hence the integral is majorized by $x^{\alpha-1}(\log N)(\log x)^2$. Since $\beta(1 - \alpha) > 2$, $x^{\alpha-1}(\log N)(\log x)^2 \ll (\log N)^{-1}$. \square

Remark 3.3. Assume that $L(s, \rho)$ satisfies GRH. Take $\alpha = 1/2 + \epsilon^2$ and $\beta = 2 + \epsilon$. Then, from the above proof, we can see that

$$\log L(1, \rho) = \sum_{n < (\log N)^{2+\epsilon}} \frac{\Lambda(n) a_\rho(n)}{n \log n} + O\left(\frac{\log \log N}{(\log N)^{\frac{\epsilon}{2} - (2\epsilon^2 + \epsilon^3)}}\right),$$

for any $\epsilon > 0$.

Now, using Proposition 3.1, we express $L(1, \rho)$ as a product over small primes. We omit p from $\alpha_i(p)$ for simplicity.

$$(3.4) \quad \sum_{n < x} \frac{\Lambda(n) a_\rho(n)}{n \log n} = \sum_{k, p^k < x} \frac{\alpha_1^k + \cdots + \alpha_d^k}{kp^k} = \sum_{p < x} \sum_{i=1}^d \sum_{k < \frac{\log x}{\log p}} \frac{1}{k} (\alpha_i p^{-1})^k.$$

Here

$$\sum_{k < \frac{\log x}{\log p}} \frac{1}{k} (\alpha_i p^{-1})^k = -\log(1 - \alpha_i p^{-1}) + A_p,$$

where

$$|A_p| \leq \sum_{k \geq \frac{\log x}{\log p}} \frac{1}{k} p^{-k} \leq \frac{\log p}{\log x} \cdot \frac{p^{-\frac{\log x}{\log p}}}{1 - p^{-1}}.$$

Here $p^{\frac{\log x}{\log p}} = x$. Hence

$$(3.4) = - \sum_{p < x} \sum_{i=1}^d \log(1 - \alpha_i p^{-1}) + d \sum_{p < x} A_p.$$

Here

$$\sum_{p < x} |A_p| \leq \frac{1}{x \log x} \sum_{p < x} \frac{\log p}{1 - p^{-1}} \leq \frac{2}{\log x}.$$

Therefore, it is summarized as follows:

Proposition 3.5. *If $L(s, \rho)$ is entire and is zero-free in the rectangle $[\alpha, 1] \times [-x, x]$, where $x = (\log N)^\beta$, $\beta(1 - \alpha) > 2$, and N is the conductor of ρ , then*

$$(3.6) \quad L(1, \rho) = \prod_{p < x} \prod_{i=1}^d (1 - \alpha_i p^{-1})^{-1} \left(1 + O\left(\frac{1}{\log x}\right) \right).$$

Furthermore, if $L(s, \rho)$ satisfies GRH, then

$$L(1, \rho) = \prod_{p < (\log N)^{2+\epsilon}} \prod_{i=1}^d (1 - \alpha_i p^{-1})^{-1} \left(1 + O\left(\frac{1}{\log \log N}\right) \right).$$

In order to find the upper and lower bound of $L(1, \rho)$, we examine the Euler product: Let C be a conjugacy class of S_{d+1} , and let C be a product of d_1, \dots, d_k cycles, where $d_i \geq 1$ for all i and $d_1 + \dots + d_k = d + 1$. Then if $Frob_p \in C$, $(1 - X) \prod_{i=1}^d (1 - \alpha_i X) = (1 - X^{d_1}) \dots (1 - X^{d_k})$. Hence

$$\prod_{i=1}^d (1 - \alpha_i p^{-1})^{-1} = (1 - p^{-1})(1 - p^{-d_1})^{-1} \dots (1 - p^{-d_k})^{-1}.$$

Now we use Mertens' theorem:

$$\prod_{p \leq y} (1 - p^{-1})^{-1} = e^\gamma (1 + o(1)) \log y.$$

Also $\prod_{p \leq y} (1 - p^{-n})^{-1} = \zeta(n) (1 + O(\frac{1}{y \log y}))$ if $n \geq 2$.

Hence the upper bound of $\prod_{i=1}^d (1 - \alpha_i p^{-1})^{-1}$ is when $C = 1$, and it is $(1 - p^{-1})^{-d}$. The lower bound is when $C = (1, \dots, d+1)$, and it is $(1 - p^{-1})(1 - p^{-d-1})^{-1}$. Moreover, it takes only the values $(1 - p^{-e_1})^{-a_1} \dots (1 - p^{-e_l})^{-a_l} (1 - p^{-1})^{a_0}$, where $e_1, \dots, e_l \geq 2$, and $-d \leq a_0 \leq 1$. Here $a_0 = 1$ only when $a_1 e_1 + \dots + a_l e_l = d + 1$. We summarize it as

$$(3.7) \quad (1 - p^{-1})(1 - p^{-d-1})^{-1} \leq \prod_{i=1}^d (1 - \alpha_i p^{-1})^{-1} \leq (1 - p^{-1})^{-d}.$$

We note that (3.7) is true even if p is ramified, i.e., when some of α_i 's are zero. Hence by the above proposition, under GRH and the strong Artin conjecture for $L(s, \rho)$, for any $\epsilon > 0$,

$$\frac{\zeta(d+1)}{(2+\epsilon)e^\gamma \log \log N} (1 + o(1)) \leq L(1, \rho) \leq (e^\gamma (2+\epsilon) \log \log N)^d (1 + o(1)).$$

Since ϵ is arbitrarily small, we showed

$$\left(\frac{1}{2} + o(1)\right) \frac{\zeta(d+1)}{e^\gamma \log \log N} \leq L(1, \rho) \leq (2 + o(1))^d (e^\gamma \log \log N)^d.$$

4. EXTREME RESIDUE VALUES

4.1. True upper and lower bound. For simplicity, we denote $L(X)^{r_2}$ by $L(X)$. Let $y = c_1 \log X$ with $c_1 > 0$. Recall that in Proposition 3.1, the conductor of $L(s, \rho)$ is $|D_K|$, and $\frac{X}{2} < |D_K| < X$, and $x = (\log X)^\beta$ for some β .

In this section we show that except for $O(X e^{-c' \frac{\log X}{\log \log X} \log \log \log X})$ in $L(X)$, the lower bound and upper bound on $L(1, \rho)$ are

$$(1 + o(1)) \frac{\zeta(d+1)}{e^\gamma (\log \log |D_K|)}, \quad (1 + o(1)) (e^\gamma \log \log |D_K|)^d, \quad \text{resp.}$$

We apply Kowalski-Michel zero density theorem [7] to the family $L(X)$. Then except for $O\left((\log X)^{\beta B} X^{(\frac{5d}{2}+1)\frac{1-\alpha}{2\alpha-1}}\right)$ L -functions, every L -function $L(s, \rho)$ in $L(X)$ is zero-free on $[\alpha, 1] \times [-(\log X)^\beta, (\log X)^\beta]$ with $\beta(1 - \alpha) > 2$. Here B is a constant depending on the family $L(X)$. We refer to [1] for the detail.

Since except for $O\left((\log X)^{\beta B} X^{(\frac{5d}{2}+1)\frac{1-\alpha}{2\alpha-1}}\right)$ L -functions, the L -functions in $L(X)$ have the desired zero-free region, we apply Proposition 3.5 to the L -functions in $L(X)$ to obtain

$$L(1, \rho) = \prod_{p < x} \prod_{i=1}^d (1 - \alpha_i p^{-1})^{-1} \left(1 + O\left(\frac{1}{\log x}\right)\right).$$

Since

$$\sum_{y < p < x} \frac{1}{p^2} \leq \sum_{p > y} \frac{1}{p^2} \leq \frac{2}{y \log y},$$

we can show

$$\prod_{y < p < x} \prod_{i=1}^d (1 - \alpha_i p^{-1})^{-1} = \exp \left(\sum_{y < p < x} \frac{a_\rho(p)}{p} \right) \left(1 + O \left(\frac{1}{y \log y} \right) \right).$$

We prove

Proposition 4.1. *Except for $O(X e^{-c' \frac{\log X}{\log \log X} \log \log \log X})$ L -functions in $L(X)$ for some constant $c' > 0$, L -functions in $L(X)$ satisfy*

$$(4.2) \quad \left| \sum_{y < p < x} \frac{a_\rho(p)}{p} \right| \leq \frac{1}{(\log \log X)^{1/2}}.$$

Hence, for L -functions which have the desired zero-free region and satisfy (4.2),

$$L(1, \rho) = \prod_{p \leq y} \prod_{i=1}^d (1 - \alpha_i p^{-1})^{-1} \left(1 + \frac{1}{(\log \log |D_K|)^{1/2}} \right).$$

This and (3.7) implies immediately Theorem 1.2.

In order to prove Proposition 4.1, we follow the idea in [8]. Namely we prove

Proposition 4.3. *Let $y = c_1 \log X$ and $r \leq c_2 \frac{\log X}{\log \log X}$ for some positive constants c_1 and c_2 . Then,*

$$\sum_{\rho \in L(X)} \left(\sum_{y < p < x} \frac{a_\rho(p)}{p} \right)^{2r} \ll 2^{2r-1} d^{2r} \frac{(2r)!}{r!} \frac{2^{2r}}{(y \log y)^r} X,$$

with an absolute implied constant.

By Stirling's formula, $2^{2r-1} d^{2r} \frac{(2r)!}{r!} \frac{2^{2r}}{(y \log y)^r} \ll \left(\frac{cd^2 r}{y \log y} \right)^r$ for a constant c .

Proof. By multinomial formula, the left hand side is

$$(4.4) \quad \sum_{\rho \in L(X)} \sum_{u=1}^{2r} \frac{1}{u!} \sum_{r_1, \dots, r_u}^{(1)} \frac{(2r)!}{r_1! \cdots r_u!} \sum_{p_1, \dots, p_u}^{(2)} \frac{a_\rho(p_1)^{r_1} \cdots a_\rho(p_u)^{r_u}}{p_1^{r_1} \cdots p_u^{r_u}},$$

where $\sum_{r_1, \dots, r_u}^{(1)}$ means the sum over the u -tuples (r_1, \dots, r_u) of positive integers such that $r_1 + \cdots + r_u = 2r$, and $\sum_{p_1, \dots, p_u}^{(2)}$ means the sum over the u -tuples (p_1, \dots, p_u) of distinct primes such

that $y < p_i < x$ for each i . Write

$$(4.4) = \sum_{u=1}^{2r} \sum_{r_1, \dots, r_u}^{(1)} \frac{(2r)!}{r_1! \cdots r_u!} \frac{1}{u!} \sum_{p_1, \dots, p_u}^{(2)} \frac{1}{p_1^{r_1} \cdots p_u^{r_u}} \left(\sum_{\rho \in L(X)} a_\rho(p_1)^{r_1} \cdots a_\rho(p_u)^{r_u} \right).$$

We will show that for any composition $r_1 + r_2 + \cdots + r_u = 2r$,

$$(4.5) \quad \frac{(2r)!}{r_1! \cdots r_u!} \frac{1}{u!} \sum_{p_1, \dots, p_u}^{(2)} \frac{1}{p_1^{r_1} \cdots p_u^{r_u}} \left(\sum_{\rho \in L(X)} a_\rho(p_1)^{r_1} \cdots a_\rho(p_u)^{r_u} \right) \ll d^{2r} X \frac{(2r)!}{r!} \frac{2^{2r}}{(y \log y)^r}.$$

Since the number of compositions of $2r$ is 2^{2r-1} , it implies that

$$(4.4) \ll 2^{2r-1} d^{2r} \frac{(2r)!}{r!} \frac{2^{2r}}{(y \log y)^r} X.$$

First, we consider compositions with $r_i \geq 2$ for all i . Then by using the trivial bound,

$$\begin{aligned} & \sum_{p_1, \dots, p_u}^{(2)} \frac{1}{p_1^{r_1} \cdots p_u^{r_u}} \left(\sum_{\rho \in L(X)} a_\rho(p_1)^{r_1} \cdots a_\rho(p_u)^{r_u} \right) \ll d^{2r} X \left(\sum_{y < p_1 < x} \frac{1}{p_1^{r_1}} \right) \cdots \left(\sum_{y < p_u < x} \frac{1}{p_u^{r_u}} \right) \\ & \ll d^{2r} X \frac{2^{2r}}{(y \log y)^r} \left(\frac{\log y}{y} \right)^{r-u}. \end{aligned}$$

Hence (4.5) is proved once we show that for any r_1, \dots, r_u such that $r_1 + \cdots + r_u = 2r$, and $r_i \geq 2$ for all i ,

$$\frac{1}{u! r_1! \cdots r_u!} \left(\frac{\log y}{y} \right)^{r-u} \leq \frac{1}{r!},$$

or equivalently

$$(4.6) \quad \frac{r!}{u! r_1! \cdots r_u!} \leq \left(\frac{y}{\log y} \right)^{r-u}.$$

Since $r_i \geq 2$ for all $i = 1, 2, \dots, u$, we have $u \leq r$. Since $y = c_1 \log X$ and $r \leq c_2 \frac{\log X}{\log \log X}$, $r \leq \frac{y}{\log y}$ for sufficiently small c_2 . Then

$$\frac{r!}{u! r_1! \cdots r_u!} \leq \frac{r!}{u!} = r(r-1) \cdots (r-u+1) \leq r^{r-u} \leq \left(\frac{y}{\log y} \right)^{r-u}.$$

Next, suppose $r_i = 1$ for some i . We may assume that $r_1 + \cdots + r_m + r_{m+1} + \cdots + r_u = 2r$, $r_1 = \dots = r_m = 1$, and $r_{m+1} > 1, \dots, r_u > 1$. First, we need a technical combinatorial lemma.

Lemma 4.7. *Let r_i 's be as above. Then*

$$(4.8) \quad \frac{1}{u!} \cdot \frac{1}{r_1! r_2! \cdots r_m! r_{m+1}! \cdots r_u!} \cdot \frac{y^u}{y^{m+r}} \cdot \frac{(\log y)^r}{(\log y)^u} \leq \frac{1}{r!}.$$

Proof. First, we assume that m is even. Then since $r_{m+1}, \dots, r_u \geq 2$, and $r_{m+1} + \dots + r_u = 2r - m$, by (4.6),

$$\frac{\left(\frac{2r-m}{2}\right)!}{(u-m)!r_{m+1}!\dots r_u!} \leq \left(\frac{y}{\log y}\right)^{(r-m/2)-(u-m)} \leq \left(\frac{y}{\log y}\right)^{r+m/2-u}$$

Hence

$$\frac{1}{r_{m+1}!\dots r_u!} \leq \frac{(u-m)!}{(r-m/2)!} \left(\frac{y}{\log y}\right)^{r+m/2-u}.$$

So

$$\begin{aligned} \frac{1}{u!} \cdot \frac{1}{r_1!r_2!\dots r_m!r_{m+1}!\dots r_u!} \frac{y^u}{y^{m+r}} \frac{(\log y)^r}{(\log y)^u} &\leq \frac{(u-m)!}{u!} \frac{1}{(r-m/2)!} \left(\frac{y}{\log y}\right)^{r+m/2-u} \frac{y^u}{y^{m+r}} \frac{(\log y)^r}{(\log y)^u} \\ &\leq \frac{(u-m)!}{u!} \frac{1}{(r-m/2)!} \frac{1}{(y \log y)^{m/2}} \end{aligned}$$

Since $r < y$ and $\frac{(u-m)!}{u!} < 1$,

$$\frac{r!}{(r-\frac{m}{2})!} \frac{(u-m)!}{u!} \leq (y \log y)^{m/2}.$$

This implies

$$\frac{(u-m)!}{u!} \frac{1}{(r-m/2)!} \frac{1}{(y \log y)^{m/2}} \leq \frac{1}{r!}.$$

Hence we have (4.8).

When m is odd, we consider a composition of $2r - m + 3$ of the form:

$$r'_{m+1} = r_{m+1}, r'_{m+2} = r_{m+2}, \dots, r'_u = r_u, \text{ and } r'_{u+1} = 3.$$

With this composition, by (4.6),

$$\frac{\left(\frac{2r-m+3}{2}\right)!}{(u-m+1)!r_{m+1}!\dots r_u!3!} = \frac{\left(\frac{2r-m+3}{2}\right)!}{(u-m+1)!r'_{m+1}!\dots r'_u!r'_{u+1}!} \leq \left(\frac{y}{\log y}\right)^{r+m/2+1/2-u}.$$

As we did for the case of even m , since $r < y$ and $\frac{(u-m+1)!}{u!} \leq 1$, we have

$$\frac{r!}{(r-\frac{m-3}{2})!} \frac{(u-m+1)!}{u!} \leq \frac{1}{6} (y \log y)^{\frac{m-1}{2}} \log y.$$

This implies (4.8). □

Recall that we are treating a composition $r_1 + r_2 + \dots + r_u = 2r$ with $r_1 = r_2 = \dots = r_m = 1$. Let N be the number of conjugacy classes of G , and partition the sum $\sum_{\rho \in L(X)}$ into $(N + w)^u$ sums, namely, given $(\mathcal{S}_1, \dots, \mathcal{S}_u)$, where \mathcal{S}_i is either $\mathcal{S}_{p_i, C}$ or \mathcal{S}_{p_i, r_j} , we consider the set of $\rho \in L(X)$ with the local conditions \mathcal{S}_i for each i . Note that in each such partition, $a_\rho(p_1)^{r_1} \dots a_\rho(p_u)^{r_u}$ remains a constant.

Suppose p_1 is unramified, and fix the splitting types of p_2, \dots, p_u , and let Frob_{p_1} runs through the conjugacy classes of G . Then by (2.2), the sum of such N partitions is

$$(4.9) \quad \sum_C \left(\frac{|C|a_\rho(p_1)}{|G|(1+f(p_1))} A(\mathcal{S}_2, \dots, \mathcal{S}_u) X + O((p_1 \cdots p_u)^\kappa X^\delta) \right),$$

for a constant $A(\mathcal{S}_2, \dots, \mathcal{S}_u)$. Let χ_ρ be the character of ρ . Then $a_\rho(p) = \chi_\rho(g)$, where $g = \text{Frob}_p$. By orthogonality of characters, $\sum_C |C|a_\rho(p_1) = \sum_{g \in G} \chi_\rho(g) = 0$. Hence the above sum is $O((p_1 \cdots p_u)^\kappa X^\delta)$. The contribution from these N partitions to (4.5) is,

$$\begin{aligned} & \ll X^\delta \frac{(2r)!}{r_1! \cdots r_u!} \frac{1}{u!} \sum_{p_1, \dots, p_u}^{(2)} p_1^{\kappa-1} \cdots p_m^{\kappa-1} p_{m+1}^{\kappa-r_{m+1}} \cdots p_u^{\kappa-r_u} \\ & \ll X^\delta \frac{(2r)!}{r_1! \cdots r_u!} \frac{1}{u!} \prod_{i=1}^m \left(\sum_{y < p_i < x} p_i^{\kappa-1} \right) \prod_{i=m+1}^u \left(\sum_{y < p_i < x} p_i^{\kappa-r_i} \right) \\ & \ll 2^u X^\delta \frac{(2r)!}{r_1! \cdots r_u!} \frac{1}{u!} \frac{x^{u\kappa}}{(\log x)^u} \ll 2^u X^\delta \frac{(2r)!}{r!} \frac{x^{u\kappa}}{(\log x)^u} y^{m+r-u} (\log y)^{u-r} \ll 2^u X^\delta \frac{(2r)!}{r!} (\log X)^{u\kappa\beta+r}. \end{aligned}$$

Here we used Lemma 4.7 for the second last inequality.

Hence the contribution from the cases when p_j is unramified for some $j \leq m$, is

$$\ll (N+w)^u 2^u X^\delta \frac{(2r)!}{r!} (\log X)^{u\kappa\beta+r} \ll (N+w)^{2r} 2^{2r} X^\delta \frac{(2r)!}{r!} (\log X)^{2r(\kappa\beta+1)}.$$

If we choose c_2 sufficiently small, for example, taking $c_2 = \frac{1-\delta}{20(\kappa\beta+1)}$,

$$(N+w)^{2r} 2^{2r} X^\delta \frac{(2r)!}{r!} (\log X)^{2r(\kappa\beta+1)} \ll d^{2r} X \frac{(2r)!}{r!} \frac{2^{2r}}{(y \log y)^r}.$$

Hence we verified (4.5).

Now, we assume that p_1, p_2, \dots, p_m are all ramified. Then by (2.2), the number of elements in the set of $\rho \in L(X)$ with the local condition $\mathcal{S}_{p_i, r}$ for $i = 1, \dots, m$, is

$$\prod_{i=1}^m \frac{f(p_i)}{1+f(p_i)} A(r_2) X + O((p_1 \cdots p_m)^\kappa X^\delta),$$

Since $\frac{f(p)}{1+f(p)} \ll \frac{1}{p}$, by the trivial bound, the main term contributes to (4.5)

$$\begin{aligned} & X d^{2r} \sum_{p_1, \dots, p_u}^{(2)} \frac{1}{p_1^2 \cdots p_m^2 p_{m+1}^{r_{m+1}} \cdots p_u^{r_u}} \ll X d^{2r} \prod_{i=1}^m \left(\sum_{y < p_i < x} p_i^{-2} \right) \prod_{i=m+1}^u \left(\sum_{y < p_i < x} p_i^{-r_i} \right) \\ & \ll X d^{2r} 2^{2r} (y \log y)^{-r} \frac{y^u}{y^{m+r}} \cdot \frac{(\log y)^r}{(\log y)^u}. \end{aligned}$$

By Lemma 4.7, (4.5) is verified.

The contribution of the error term $O((p_1 \cdots p_m)^\kappa X^\delta)$ is the same as when p_1 is unramified. \square

Now take $y = c_1 \log X$, and $r = c_2 \frac{\log x}{\log \log X}$. Then from Proposition 4.3, the number of $\rho \in L(X)$ such that $\left| \sum_{y < p < x} \frac{a_\rho(p)}{p} \right| > \frac{1}{(\log \log X)^{1/2}}$, is

$$(4.10) \quad \ll X e^{-c' \frac{\log X}{\log \log X} \log \log \log X},$$

for some $c' > 0$. This proves Proposition 4.1.

4.2. Infinite family of number fields with extreme residues. Let C be a conjugacy class of S_{d+1} , and $\mathcal{S} = (S_{p,C})_{p \leq y}$ be the set of local conditions such that for every prime $p \leq y$, $Frob_p \in C$. We denote $L(X, \mathcal{S})^{r_2}$ by $L(X, \mathcal{S})$. Conjecture 2.1 says that

$$|L(X, \mathcal{S})| = A(r_2) X \prod_{p \leq y} \frac{\frac{|C|}{|S_{d+1}|}}{1 + f(p)} + O \left(\left(\prod_{p \leq y} p \right)^\gamma X^\delta \right).$$

The main term is

$$(4.11) \quad A(r_2) \frac{X}{\log y} \exp \left(-\log \frac{|S_{d+1}|}{|C|} \cdot \frac{\log X}{\log \log X} \right).$$

This is larger than (4.10). Also we may assume that almost all L -functions in $L(X, \mathcal{S})$ have the desired zero-free region of the form in Proposition 3.5. Hence, by Proposition 4.1, except $O(X e^{-c' \frac{\log X}{\log \log X} \log \log \log X})$ fields,

$$L(1, \rho) = \prod_{\substack{p \leq y \\ Frob_p \in C}} \prod_{i=1}^d (1 - \alpha_i p^{-1})^{-1} \left(1 + O \left(\frac{1}{(\log \log |D_K|^{\frac{1}{2}})} \right) \right).$$

By taking $C = 1$, we obtain an infinite family of number fields with the upper bound. On the other hand, by taking $C = (1, \dots, d+1)$, we obtain an infinite family of number fields with the lower bound. This proves Theorem 1.3.

In a similar way, for each $0 \leq i \leq d$, $d-i$ even, we can construct an infinite family of number fields with the residue

$$\zeta(2)^{\frac{d-i}{2}} e^{\gamma i} (\log \log |D_K|)^i (1 + o(1)).$$

In particular we obtain an infinite family of number fields with bounded residues by taking

$$C = \begin{cases} (1, 2)(3, 4) \cdots (d-1, d), & \text{if } d \text{ is even} \\ (1, 2)(3, 4) \cdots (d-4, d-3)(d-2, d-1, d), & \text{if } d \text{ is odd} \end{cases}.$$

for which

$$Res_{s=1}\zeta_K(s) = L(1, \rho) = \begin{cases} \zeta(2)^{\frac{d}{2}}(1 + o(1)), & \text{if } d \text{ is even} \\ \zeta(2)^{\frac{d-3}{2}}\zeta(3)(1 + o(1)), & \text{if } d \geq 3 \text{ is odd.} \end{cases},$$

and it proves Theorem 1.4.

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